Tutorial 1 (16 Sep)

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Foreword

⓪ Tutorial notes will be uploaded to the course webpage after tutorials.

Also, tutorials will be recorded (together with Wednesday's lectures).

① Personal Information:

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② Tutorial Arrangement (16 Sep - 21 Oct):

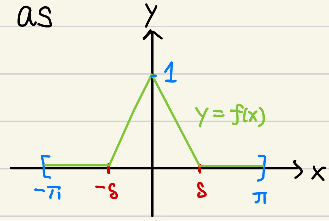
- (0830-0915) Problems and Solutions
- (0915-0930) Q and A for HW problems

(Optional; effective from 23 Sep)

③ Reference: · (Fourier Series): Stein-Shakarchi

· (Metric Spaces): Rudin; Copson

Q1) Given $0<\delta<\pi$, define $f:[-\pi,\pi]\longrightarrow\mathbb{R}$ as



$$f(x)=\begin{cases}0 & , |x|\geq\delta \\ 1-\frac{|x|}{\delta} & , |x|\leq\delta\end{cases}$$

Compute its real Fourier series.

Sol) Idea : Direct computation by definition.

Recall that the real Fourier series of f is $a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$, where

- $a_0 := \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx = \frac{1}{2\pi}\cdot\text{Area}(\triangle) = \frac{1}{2\pi}\cdot\left(\frac{1}{2}\cdot2\delta\cdot1\right)=\frac{\delta}{2\pi}$.

- $\forall n>0$, $a_n := \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$ ($\because$ $f(x)\cos nx$ is an even function.)

$$=\frac{2}{\pi}\int_0^{\delta}\left(1-\frac{x}{\delta}\right)\cos nx\,dx=\frac{2}{n\pi}\int_0^{\delta}\left(1-\frac{x}{\delta}\right)d(\sin nx)$$

$$=\frac{2}{n\pi}\left\{\left[\left(1-\frac{x}{\delta}\right)(\sin nx)\right]_0^{\delta}-\int_0^{\delta}(\sin nx)\cdot\left(-\frac{1}{\delta}\right)dx\right\}$$

$$=\frac{2}{n\pi}\left\{0+\frac{1}{\delta}\int_0^{\delta}\sin nx\,dx\right\}$$

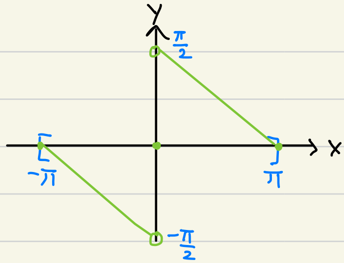
$$=\frac{2}{n\pi\delta}\left[-\frac{\cos nx}{n}\right]_0^{\delta}=\frac{2(1-\cos n\delta)}{n^2\pi\delta}$$

- $b_n := \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx = 0$ ($\because$ $f(x)\sin nx$ is an odd function.)

Therefore, the real Fourier series of f is $\frac{\delta}{2\pi}+2\sum_{n=1}^{\infty}\frac{1-\cos n\delta}{n^2\pi\delta}\cos nx$.

Q2) Define $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & -\pi \leq x < 0 \\ 0, & x = 0 \\ \frac{\pi}{2} - \frac{x}{2}, & 0 < x \leq \pi \end{cases}$$



(a) Compute its complex Fourier series $S(f)(x) := \sum_{n=-\infty}^{+\infty} c_n e^{inx}$.

(b) Show that $S(f)$ converges pointwisely on $[-\pi, \pi]$, i.e. $\lim_{m\to\infty}\left(c_0 + \sum_{n=1}^{m}\left(c_n e^{inx} + c_{-n}e^{-inx}\right)\right)$ exists, $\forall x \in [-\pi, \pi]$.

Sol) (a) Idea : Direct computation by definition.

Recall that the complex Fourier coefficients c_n are given by

- $c_0 := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$. ($\because$ $f(x)$ is an odd function.)

- $\forall n \neq 0,\ c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0}\left(-\frac{\pi}{2} - \frac{x}{2}\right)e^{-inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - \frac{x}{2}\right)e^{-inx}\,dx$

$$= \frac{1}{4}\left(\int_{-\pi}^{0} -e^{-inx}\,dx + \int_{0}^{\pi} e^{-inx}\,dx\right) - \frac{1}{4\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx$$

$$= \frac{1}{4}\left(\left[\frac{e^{-inx}}{in}\right]_{-\pi}^{0} + \left[-\frac{e^{-inx}}{in}\right]_{0}^{\pi}\right) + \frac{1}{4n\pi i}\int_{-\pi}^{\pi} x\,d(e^{-inx})$$

$$= \frac{1}{4ni}\left(\left[1-(-1)^n\right] + \left[-(-1)^n + 1\right]\right) + \frac{1}{4n\pi i}\left(\left[xe^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx}\,dx\right)$$

$$= \frac{1}{2ni}\left(1-(-1)^n\right) + \frac{1}{4n\pi i}\left((-1)^n\cdot 2\pi - \left[-\frac{e^{-inx}}{in}\right]_{-\pi}^{\pi}\right) = \frac{1}{2ni}\left(1-(-1)^n + (-1)^n\right) = \frac{1}{2ni}$$

Therefore, the complex Fourier series of f is $S(f)(x) = \frac{1}{2i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\cdot e^{inx}$.

(b) Idea: Simplify the series and then apply a suitable convergence test.

Note that $\forall m \in \mathbb{N}$, $\frac{1}{2i}\sum_{n=1}^{m}\left(\frac{1}{n}e^{inx} - \frac{1}{n}e^{-inx}\right) = \sum_{n=1}^{m}\frac{1}{n}\left(\frac{e^{inx}-e^{-inx}}{2i}\right) = \sum_{n=1}^{m}\frac{\sin nx}{n}$.

Recall Dirichlet test (See e.g. Bartle's "Introduction to Real Analysis", 9.3.4):

Prop If the sequences of real numbers $\{x_n\}, \{y_n\}$ satisfy

- $\{x_n\}$: decreasing with $\lim_{n\to\infty} x_n = 0$.
- $\left\{\sum_{n=1}^{m} y_n\right\}_{m=1}^{\infty}$: bounded.

then $\lim_{m\to\infty}\sum_{n=1}^{m} x_n y_n$ exists. — $\blacksquare$

$\forall x \in [-\pi, \pi]$, apply the test with $x_n = \frac{1}{n}$ and $y_n = \sin nx$.

- $\{\frac{1}{n}\}$: decreasing with $\lim_{n\to\infty}\frac{1}{n} = 0$.
- For $x=0$, $\sum_{n=1}^{m}\sin nx = 0$; For $x \neq 0$, $\sum_{n=1}^{m}\sin nx = \operatorname{Im}\left(\sum_{n=1}^{m} e^{inx}\right) = \operatorname{Im}\left(e^{ix}\left(\frac{1-e^{imx}}{1-e^{ix}}\right)\right)$

$$\Rightarrow \left|\sum_{n=1}^{m}\sin nx\right| \leq \left|e^{ix}\left(\frac{1-e^{imx}}{1-e^{ix}}\right)\right| \leq 1\cdot\frac{1+1}{|1-e^{ix}|} = \frac{2}{|1-e^{ix}|} \text{ is bounded.}$$

$\therefore$ By Dirichlet test, $S(f)$ converges pointwisely on $[-\pi, \pi]$.

Rmk This question provides an example of discontinuous function

with its Fourier series converges pointwisely, but NOT uniformly on $[-\pi, \pi]$ (Why?).

Q3) Given two 2π-periodic continuous functions $f, g : \mathbb{R} \to \mathbb{C}$, their convolution product $f * g : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic continuous function defined as $(f*g)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy$

(a) Simplify $f * D_n$, where $n \in \mathbb{N}$ and $D_n(y) := \sum_{k=-n}^{n} e^{iky}$.

(b) Show that for any $k \in \mathbb{Z}$, $c_k(f*g) = c_k(f) \cdot c_k(g)$, where $c_k(f)$ (resp. $c_k(g)$, $c_k(f*g)$) is the k^{th} (complex) Fourier coeffient of f (resp. g, $f*g$).

Sol) (a) Idea: Direct computation by definition.

$$\begin{aligned}
(f * D_n)(x) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) \sum_{k=-n}^{n} e^{iky}\, dy \\
&= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, e^{iky}\, dy \\
&= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(z)\, e^{ik(x-z)}\, d(-z) \quad (\text{apply a change of variable } z = x-y) \\
&= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(z)\, e^{-ikz}\, dz \right) e^{ikx} \\
&= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(z)\, e^{-ikz}\, dz \right) e^{ikx} \quad (\because f(z)\, e^{-ikz} \text{ is } 2\pi\text{-periodic}) \\
&= \sum_{k=-n}^{n} c_k(f)\, e^{ikx} = S_n(f)(x), \text{ the value of } n^{th} \text{ partial sum of } S(f) \text{ at } x.
\end{aligned}$$

Rmk This computation is useful in studying the pointwise convergence of $S(f)$ when f is Lipschitz continuous.

(b) Idea: Apply Fubini's Theorem to interchange the order of integration.

$$\text{LHS} = c_k(f*g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)\,e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy\right)e^{-ikx}\,dx$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,e^{-ik(x-y)}e^{-iky}\,dy\,dx$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x-y)\,e^{-ik(x-y)}\,dx\right)g(y)\,e^{-iky}\,dy$$

($\because$ Fubini's Theorem on continuous function $F_k(x,y) = f(x-y)g(y)e^{-ikx} : [-\pi,\pi] \to \mathbb{C}$)

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi-y}^{\pi-y} f(z)\,e^{-ikz}\,dz\right)g(y)\,e^{-iky}\,dy$$

(apply change of variables $z = x-y$)

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(z)\,e^{-ikz}\,dz\right)g(y)\,e^{-iky}\,dy$$

($\because$ $f(z)e^{-ikz}$ is 2π-periodic)

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)\,e^{-ikz}\,dz\right)\cdot\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\,e^{-iky}\,dy\right)$$

$$= c_k(f)\cdot c_k(g) = \text{RHS}$$

Rmk
- (b) shows that the convolution product of functions corresponds to the pointwise product of their Fourier coefficients as functions on $\mathbb{Z}$.
- (b) holds more generally: for f, g being 2π-periodic integrable only.

Idea of proof: approximate f, g by sequences of continuous functions $\{f_n\}, \{g_n\}$ and apply above result to $f_n * g_n$, then argue that $c_k(f_n) \to c_k(f)$ as $n \to \infty$.